

Nonadiabatic holonomic single-qubit gates in off-resonant Λ systems

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Abstract

We generalize nonadiabatic holonomic quantum computation in a resonant Λ configuration proposed in [New J. Phys. 14 (2012) 103035] to the case of off-resonant driving lasers. We show that any single-qubit holonomic gate can be realized by separately varying the detuning, amplitude, and phase of the lasers.

Key words: Geometric phase; Quantum gates

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Holonomic quantum computation (HQC) is the idea to use non-Abelian geometric phases to implement a universal set of quantum gates. It was first proposed in the context of adiabatic evolution by Zanardi and Rasetti [1] based on Wilczek-Zee geometric phases [2] associated with degenerate energy subspaces driven by slowly varying parameters. More recently, a scheme for fast holonomic quantum gates has been proposed [3]. This scheme has subsequently been implemented experimentally [4,5,6,7].

The nonadiabatic HQC scheme in Ref. [3] is based on a Λ system driven by short resonant laser pulses. The high-speed feature makes the resulting gates potentially easier to implement as it implies a shorter exposure to detrimental decoherence effects. Here, we modify the original setup in Ref. [3] by allowing nonvanishing detuning of the two lasers driving off-resonant transitions between the excited state and the computational levels. This modified scheme can be used to implement any single-qubit gate by separately varying the detuning, amplitude, and phase of the lasers, at the expense of restricting to square-shaped pulses.

Consider a quantum system exhibiting a three-level Λ -type configuration, in which two energy levels $|0\rangle$ and $|1\rangle$, spanning the computational state space,

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are coupled to an excited state $|e\rangle$ by two pulsed laser beams with detuning δ and associated with Rabi frequencies $f_0(t)$ and $f_1(t)$, see Fig. 1. The Hamiltonian in a frame that rotates with the laser fields reads ($\hbar = 1$ from now on)

$$\begin{aligned} H(t) &= F(t) \left(e^{-i\varphi} \sin \frac{\theta}{2} |e\rangle \langle 0| - \cos \frac{\theta}{2} |e\rangle \langle 1| + \text{H.c.} \right) + \delta |e\rangle \langle e| \\ &\equiv F(t) H_0 + \delta |e\rangle \langle e|, \end{aligned} \quad (1)$$

where rapidly oscillating counter-rotating terms have been neglected (rotating wave approximation) and we have put $f_0(t) = F(t)e^{-i\varphi} \sin \frac{\theta}{2}$ and $f_1(t) = -F(t) \cos \frac{\theta}{2}$. Here, θ and φ are time independent over the duration of the pulse pair, which is controlled by the real-valued pulse envelope $F(t)$.

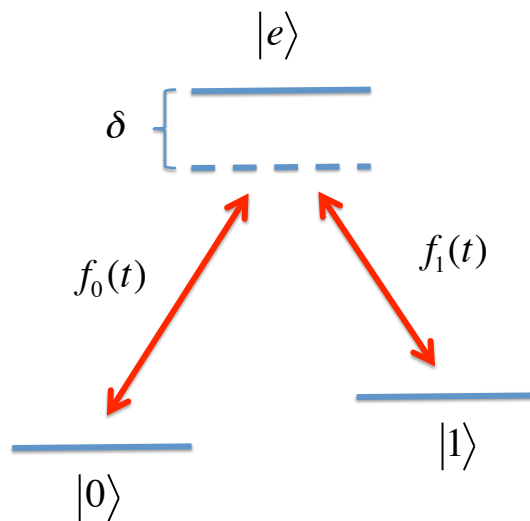


Fig. 1. Off-resonant Λ system consisting of two energy levels $|0\rangle$ and $|1\rangle$ coupled to an excited state $|e\rangle$ by two pulsed laser beams with detuning δ and associated with Rabi frequencies $f_0(t)$ and $f_1(t)$.

The standard form of nonadiabatic HQC in the Λ configuration [3] assumes that the lasers are on resonance with the transition frequencies, i.e., that the detuning δ vanishes. In this case, the evolution of the computational subspace $\mathcal{M} = \text{Span}\{|0\rangle, |1\rangle\}$ becomes purely geometric and cyclic with period τ such that $\int_0^\tau F(t)dt = \pi$, irrespective of the detailed form of $F(t)$. The path $C_{\mathbf{n}}$ traversed by the computational subspace in the Grassmannian $G(3;2)$ ¹ is

¹ That is, the space of two-dimensional subspaces of a three-dimensional complex

parametrized by the fixed laser parameters θ and φ , as captured by the unit vector $\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. Since the evolution is purely geometric, the holonomic one-qubit gate $U(C_{\mathbf{n}})$ associated with $C_{\mathbf{n}}$ coincides with the action of the time evolution operator $U(\tau, 0)$ on \mathcal{M} , i.e., $P_{\mathcal{M}}U(\tau, 0)P_{\mathcal{M}}$ with $P_{\mathcal{M}} = |0\rangle\langle 0| + |1\rangle\langle 1|$. Explicitly, we find [3]

$$U(C_{\mathbf{n}}) = |d\rangle\langle d| - |b\rangle\langle b| \quad (2)$$

with the dark and bright states $|d\rangle = \cos \frac{\theta}{2} |0\rangle + e^{-i\varphi} \sin \frac{\theta}{2} |1\rangle$ and $|b\rangle = e^{i\varphi} \sin \frac{\theta}{2} |0\rangle - \cos \frac{\theta}{2} |1\rangle$, respectively, which constitute another orthonormal frame spanning \mathcal{M} . We may write

$$U(C_{\mathbf{n}}) = ie^{-i\frac{1}{2}\pi\mathbf{n}\cdot\boldsymbol{\sigma}} = \mathbf{n} \cdot \boldsymbol{\sigma} \quad (3)$$

with $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the Pauli operators $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$, and $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$. Here, $U(C_{\mathbf{n}})$ is the nonadiabatic non-Abelian geometric phase [8] associated with the path $C_{\mathbf{n}}$.

$U(C_{\mathbf{n}})$ corresponds to a π rotation of the qubit around the direction \mathbf{n} . To obtain an arbitrary SU(2) operation, this gate must therefore be combined with another holonomic gate produced by a second pulse pair. To see this, assume that the laser parameters of two sequentially applied pulse pairs define unit vectors \mathbf{n} and \mathbf{m} . The combined gate reads

$$U(C_{\mathbf{m}})U(C_{\mathbf{n}}) = \mathbf{m} \cdot \mathbf{n} - i\boldsymbol{\sigma} \cdot (\mathbf{n} \times \mathbf{m}). \quad (4)$$

This corresponds to a rotation angle $2 \arccos(\mathbf{n} \cdot \mathbf{m})$ around the rotation axis $\mathbf{n} \times \mathbf{m}$, i.e., an arbitrary SU(2).

Let us now turn to the off-resonant case where $\delta \neq 0$. Here, the geometric nature of the evolution of \mathcal{M} depends on the detailed form of the pulse envelope $F(t)$ since $H(t)$ may no longer commute with the time evolution operator². However, there is one physically justified choice where the evolution of \mathcal{M} is purely geometric, viz., when $F(t)$ is a square pulse, i.e., $F(t) = F_0$ for $0 \leq t \leq \tau$ and zero otherwise. For such a pulse, consider evolution between t_0

vector space.

² The geometric nature depends on the pulse form since the extra term $\delta|e\rangle\langle e|$ in the Hamiltonian $H(t)$ makes it in general necessary to use time ordering to compute the time evolution operator. As a consequence, $H(t)$ does not necessarily commute with the associated time evolution operator, which in general implies that $U^\dagger(t, 0)H(t)U(t, 0)$ would not vanish on \mathcal{M} and thereby creating a nontrivial non-Abelian dynamical phase acting on the qubit. The square pulse takes care of this since the time evolution operator acts trivially on \mathcal{M} before and after the pulse and can be computed without time ordering during the pulse.

and t_1 , where $t_0 \leq 0$ and $t_1 \geq \tau$. The corresponding time evolution operator reads:

$$U(t_1, t_0) = U(t_1, \tau)U(\tau, 0)U(0, t_0) = e^{-i(t_1-\tau)\delta|e\rangle\langle e|}U(\tau, 0)e^{it_0\delta|e\rangle\langle e|} \quad (5)$$

with

$$U(\tau, 0) = e^{-i\tau(F_0H_0 + \delta|e\rangle\langle e|)}. \quad (6)$$

The action of $U(t_1, t_0)$ is trivial on \mathcal{M} on $t_0 < t < 0$ and $\tau < t < t_1$, provided τ is chosen such that \mathcal{M} undergoes cyclic evolution. Thus, it is sufficient to consider $U(\tau, 0)$ in the following.

Since $\langle k|U^\dagger(t, 0)(F_0H_0 + \delta|e\rangle\langle e|)U(t, 0)|l\rangle = \langle k|(F_0H_0 + \delta|e\rangle\langle e|)|l\rangle = 0$, $k, l = 0, 1$, on $0 \leq t \leq \tau$, the nontrivial part $U(\tau, 0)$ of the time evolution operator is purely geometric on the single-qubit subspace \mathcal{M} . It further corresponds to cyclic evolution with period $\tau = 2\pi/\sqrt{\delta^2 + 4F_0^2}$ for which we find holonomic gate

$$U(\mathbf{n}, \chi) = |d\rangle\langle d| - e^{-i\chi}|b\rangle\langle b| = e^{i\frac{1}{2}(\pi-\chi)}e^{-i\frac{1}{2}(\pi-\chi)\mathbf{n}\cdot\boldsymbol{\sigma}} \quad (7)$$

with

$$\chi = \frac{\pi\delta}{\sqrt{\delta^2 + 4F_0^2}}. \quad (8)$$

Up to the unimportant overall phase factor $e^{i\frac{1}{2}(\pi-\chi)}$, we see that $U(\mathbf{n}, \chi)$ corresponds to a single-qubit rotation with angle $\pi - \chi$ around the direction \mathbf{n} . Thus, an arbitrary single-qubit operation can be reached by independently varying the detuning and laser parameters θ and φ . In particular, $U(\mathbf{n}, \chi)$ connects to the identity in the $\delta/(2F_0) \rightarrow \infty$ limit³, i.e., $U(\mathbf{n}, \chi \rightarrow \pi) = \hat{1}$, and it coincides with the nonadiabatic holonomic gate proposed in Ref. [3] in the resonant case, i.e., $U(\mathbf{n}, 0) = U(C_{\mathbf{n}})$.

Further insight into the geometry of $U(\mathbf{n}, \chi)$ can be obtained by calculating the connection 1-form associated with the time-evolved computational sub-

³ In the small rotation angle limit, δ should thus be very large compared to the pulse strength F_0 . By comparing with Eq. (8), we see that the duration τ becomes very short in this limit, which may invalidate the rotating wave approximation. This in turn implies that the geometric nature as well as the stability of the gate would be lost [9]. Thus, gates corresponding to small qubit rotations are probably difficult to realize in practise by using our scheme.

space, i.e., $\mathcal{M}(t) = \text{Span}\{U(t, 0) |d\rangle, U(t, 0) |b\rangle\}$ with $\mathcal{M}(\tau) = \mathcal{M}(0) = \mathcal{M}$. By solving the Schrödinger equation, we find

$$\begin{aligned} |d(t)\rangle &= U(t, 0) |d\rangle = |d\rangle, \\ |b(t)\rangle &= U(t, 0) |b\rangle \\ &= e^{-i\frac{1}{2}\delta t} \left(e^{-i\frac{1}{2}\sqrt{\delta^2 + 4F_0^2}t} \sin \nu |+\rangle + e^{i\frac{1}{2}\sqrt{\delta^2 + 4F_0^2}t} \cos \nu |-\rangle \right), \end{aligned} \quad (9)$$

where

$$\tan \nu = \frac{\delta + \sqrt{\delta^2 + 4F_0^2}}{2F_0} \quad (10)$$

and $|\pm\rangle$ are the bright eigenstates of $H_0 + \delta |e\rangle \langle e|$. Clearly, the only nonzero component of the vector potential

$$A(t) = i \sum_{k,l=d,b} \langle k(t) | \dot{l}(t) \rangle |k(t)\rangle \langle l(t)| = \sum_{k,l=d,b} A_{kl}(t) |k(t)\rangle \langle l(t)| \quad (11)$$

is $A_{bb}(t)$. We find,

$$A_{bb}(t) = -\sqrt{\delta^2 + 4F_0^2} \sin^2 \nu. \quad (12)$$

The holonomy can be obtained as $|d\rangle \langle d| + e^{i\gamma} |b\rangle \langle b|$, where γ is the Aharonov-Anandan geometric phase [10] associated with $|b(t)\rangle$, i.e.,

$$\begin{aligned} \gamma &= \int_0^\tau A_{bb}(t) dt = -2\pi \sin^2 \nu = -2\pi \frac{1}{2} \left(1 + \frac{\delta}{\sqrt{\delta^2 + 4F_0^2}} \right) \\ &= \pi - \chi, \text{ mod}(2\pi), \end{aligned} \quad (13)$$

where we have used Eq. (10). Thus, $|d\rangle \langle d| + e^{i\gamma} |b\rangle \langle b|$ coincides with $U(\mathbf{n}, \chi)$.

Note that $A(t)$ commutes with itself within each pulse pair. This is the underlying reason why the holonomy $U(\mathbf{n}, \chi)$ can be understood in terms of the Abelian geometric phase factor $e^{i\gamma}$ of $|b(t)\rangle$. However, $[A(t), \tilde{A}(t')]$ can be nonvanishing for A and \tilde{A} evaluated for two subsequent pulse pairs with different laser parameters. To see this, consider $A(t) = A_{bb}(t) |b(t)\rangle \langle b(t)|$ and $\tilde{A}(t) = A_{\tilde{b}\tilde{b}}(t) |\tilde{b}(t)\rangle \langle \tilde{b}(t)|$ with $|b(t)\rangle$ and $|\tilde{b}(t)\rangle$ corresponding to two different sets of laser parameters θ, φ, δ and $\tilde{\theta}, \tilde{\varphi}, \tilde{\delta}$, respectively. We obtain

$$\begin{aligned} [A(t), \tilde{A}(t')] &= A_{bb}(t) A_{\tilde{b}\tilde{b}}(t') \\ &\times \left(|b(t)\rangle \langle b(t)| \tilde{b}(t) \rangle \langle \tilde{b}(t)| - |\tilde{b}(t)\rangle \langle \tilde{b}(t)| b(t) \rangle \langle b(t)| \right) \neq 0, \end{aligned} \quad (14)$$

where t and t' belong to the support of the respective two laser pulse pairs. This proves the non-Abelian nature of the gate.

Before concluding, let us briefly comment on how the restriction to square-shaped pulses may influence the flexibility of our scheme. The key point here is that it should be possible to vary freely the detuning, amplitude, and phase of the pulses, despite this restriction. To see explicitly what this means, let us consider a possible implementation of our scheme in which transitions between two atomic levels $j = 0, 1$ and an excited state e are induced by pulsed electric fields $\mathbf{E}_j(t) = g_j(t) \cos(\omega_j t) \boldsymbol{\epsilon}_j$, where $\boldsymbol{\epsilon}_j$ are the polarizations. Here, the time dependent part consists of two factors: $g_j(t)$ determining the pulse shape and $\cos(\omega_j t)$ determining the detuning $\delta_j = \omega_{je} - \omega_j$, ω_{je} being the energy difference (in units where $\hbar = 1$) between the bare energy eigenstates $|e\rangle$ and $|j\rangle$. The scheme requires that $g_0(t) = g_1(t) \propto F(t)$ being square-shaped and $\delta_0 = \delta_1 = \delta$. The amplitude and phase parameters θ and φ are determined by the ratio $\langle e | \boldsymbol{\mu} \cdot \boldsymbol{\epsilon}_0 | 0 \rangle / \langle e | \boldsymbol{\mu} \cdot \boldsymbol{\epsilon}_1 | 1 \rangle = -e^{-i\varphi} \tan \frac{\theta}{2}$, $\boldsymbol{\mu}$ being the electric dipole operator. We thus see that the required flexibility is obtained in this particular setting if the polarization $\boldsymbol{\epsilon}_j$ and oscillation frequency ω_j of each electric field pulse can be varied independently, although $g_j(t)$ is restricted to having square shape.

In conclusion, we have demonstrated holonomic single-qubit gates in off-resonant Λ system. These gates can be used to implement any single-qubit gate and would, together with an entangling holonomic two-qubit gate, constitute an all-geometric universal set of gates. The off-resonant holonomic gates require square-shaped pulses in order to preserve the purely geometric nature. Our finding implies that the assumption of zero detuning in the original scheme [3] is not a necessary requirement to perform nonadiabatic holonomic quantum computation in Λ systems. We further note that the additional flexibility associated with the detuning makes it possible to perform arbitrary single-qubit operations by a single pulse pair. This latter feature may help experimental realizations of holonomic quantum computation as it reduces the number of pulses needed to implement arbitrary single-qubit operations. The scheme can be implemented experimentally in various systems, such as trapped atoms or ions, superconducting qubits, or NV-centers in diamond.

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